Problem Set 2 - Solution - LV 141.A55 QISS

1. Qubit States

(a)

$\lvert\Psi\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle - i\lvert 1\rangle)$	$\theta = \pi/2$, $\varphi = \pi/2$
$\lvert\Psi\rangle = \frac{1}{4}\lvert 0\rangle + \frac{3}{4}\lvert 1\rangle$	$\langle\Psi\vert\Psi\rangle \neq 1$ not a valid state
$\lvert\Psi\rangle = \frac{1}{\sqrt{2}}(i\lvert 0\rangle + \lvert 1\rangle)$	This state is the same as the first state up to a irrelevant global phase
$\lvert\Psi\rangle = -0.6\lvert 0\rangle + 0.8\lvert 1\rangle$	$\theta = 1.8546$, $\varphi = \pi$
$\lvert\Psi\rangle = \sqrt{\frac{1}{2}}\sqrt{1+\frac{1}{\sqrt{3}}}\lvert 0\rangle + \frac{1+i}{2}\sqrt{1-\frac{1}{\sqrt{3}}}\lvert 1\rangle$	$\theta = 0.9553 = \arccos(\frac{1}{\sqrt{3}})$, $\varphi = -\pi/4$

In Python you can verify the norm with the following command

```
psi=array([[-0.6],[0.8]])
dot(transpose(psi),psi)
```

The angles can be calculated in the following way

```
from pauli import *
sx=dot(transpose(psi),dot(sigma_x,psi))
sy=dot(transpose(psi),dot(sigma_y,psi))
sz=dot(transpose(psi),dot(sigma_z,psi))
theta=arccos(sz)
phi=arctan2(sy,sx)
```

(b)

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathrm{tr}(\rho) = 2$, not a valid density matrix
$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$\mathrm{tr}(\rho^2) = 1$ pure state, (1,0,0)
$\begin{pmatrix} \frac{3}{4} & \frac{-i}{4} \\ \frac{i}{4} & \frac{1}{4} \end{pmatrix}$	$\mathrm{tr}(\rho^2) = 3/4$ mixed state, $(0,\frac{1}{2},\frac{1}{2})$
$\begin{pmatrix} \frac{1}{2} & \frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1}{2} \end{pmatrix}$	$\mathrm{tr}(\rho^2) = 3/2$, not a valid density matrix
$\begin{pmatrix} \frac{3+\sqrt{3}}{6} & \frac{1+i}{2\sqrt{3}} \\ \frac{1-i}{2\sqrt{3}} & \frac{3-\sqrt{3}}{6} \end{pmatrix}$	$\mathrm{tr}(\rho^2) = 1$ pure state, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ note: this is the same state as the last in exercise 1(a) $\lvert\Psi\rangle\langle\Psi\rvert$
$\begin{pmatrix} \frac{1}{2} & \frac{-1+i}{2\sqrt{2}} \\ \frac{-1-i}{2\sqrt{2}} & \frac{1}{2} \end{pmatrix}$	$\mathrm{tr}(\rho^2) = 1$ pure state, $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$
$\begin{pmatrix} 0.9607 & 0.1661 - 0.1006i \\ 0.1661 + 0.1006i & 0.0393 \end{pmatrix}$	$\mathrm{tr}(\rho^2) = 1$ pure state, (0.3322, 0.2012, 0.9214)

With Python you solve the problem in the following way

```
rho=array([[0.9607,0.1661-0.1006j],[0.1661+0.1006j, 0.0393]])
trace(rho)
trace(dot(rho,rho))
trace(dot(rho,sigma_x))
trace(dot(rho,sigma_y))
trace(dot(rho,sigma_z))
```

2. Qubit Hamiltonian

$$H = A\sigma_z + D\sigma_x = \begin{pmatrix} A & D \\ D & -A \end{pmatrix}$$

Eigenenergies

$$E_{0,1} = \mp\sqrt{A^2 + D^2}$$

We can introduce the angle δ, which is the angle between the original states $|0\rangle$, $|1\rangle$ and the eigenstates $|\xi_0\rangle$, $|\xi_1\rangle$ (see Figure 1)

$$\tan(\delta) = \frac{D}{A}$$

Eigenstates can now easily expressed in terms of a rotation with angle δ

$$\begin{aligned} |\xi_0\rangle &= \cos(\delta/2)\,|0\rangle - \sin(\delta/2)\,|1\rangle \\ |\xi_1\rangle &= \sin(\delta/2)\,|0\rangle + \cos(\delta/2)\,|1\rangle \end{aligned}$$

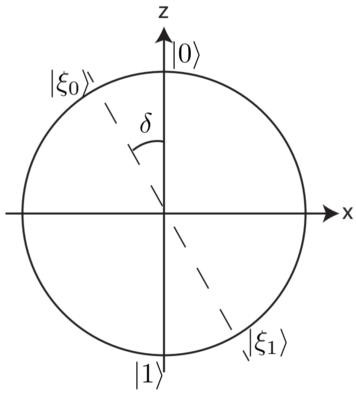


Figure 1: Eigenstates ξ_0 and ξ_1 on the Bloch Sphere. The angle δ describes the angle between the original states $|0\rangle$, $|1\rangle$ and the eigenstates $|\xi_0\rangle$, $|\xi_1\rangle$.

Since the Hamiltonian is time constant, you can easily calculate the time evolution

$$|\Psi(t)\rangle = e^{iE_0 t/\hbar}\,|\xi_0\rangle\,\langle\xi_0|\Psi(0)\rangle + e^{iE_1 t/\hbar}\,|\xi_1\rangle\,\langle\xi_1|\Psi(0)\rangle$$

3. Rabi's Formula

(a) Plugging the following Hamiltonian

$$H = \frac{\hbar\omega_0}{2}\sigma_z + \frac{A}{2}\left(\sigma_x \cos(\omega t) + \sigma_y \sin(\omega t)\right)$$

into the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$$

and introducing $\hbar\omega_1 := A$ and $|\Psi(t)\rangle = a_1(t)\,|1\rangle + a_0(t)\,|0\rangle$, we end up with the following coupled differential equation system

$$i\frac{\partial}{\partial t}\begin{pmatrix} a_1(t) \\ a_0(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} a_1(t) \\ a_0(t) \end{pmatrix}$$

The problem now is that we have time varying coefficients ($e^{i\omega t}$) in the off diagonal terms. We can get rid of the terms by making a transformation into a rotating frame:

$$\begin{aligned} b_1(t) &= a_1(t)e^{i\omega t/2} \\ b_0(t) &= a_0(t)e^{-i\omega t/2} \end{aligned}$$

$$\left|\tilde{\Psi}(t)\right\rangle = b_1(t)\,|1\rangle + b_0(t)\,|0\rangle$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} b_1(t) \\ b_0(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{pmatrix}\begin{pmatrix} b_1(t) \\ b_0(t) \end{pmatrix}$$

where $\Delta\omega = \omega_0 - \omega$.

(b) We can apply the solution from problem 2:

$$\langle 1|\Psi(t)\rangle = \left\langle 1|\tilde{\Psi}(t)\right\rangle = e^{iE_0t/\hbar}\langle 1|\xi_0\rangle\langle\xi_0|0\rangle + e^{iE_1t/\hbar}\langle 1|\xi_1\rangle\langle\xi_1|0\rangle$$

The eigenenergies are

$$\begin{aligned} E_1 &= \hbar\sqrt{\Delta\omega^2+\omega_1^2} \\ E_0 &= -\hbar\sqrt{\Delta\omega^2+\omega_1^2} \end{aligned}$$

and the rotation angle

$$\tan\delta = \frac{\omega_1}{\Delta\omega}$$

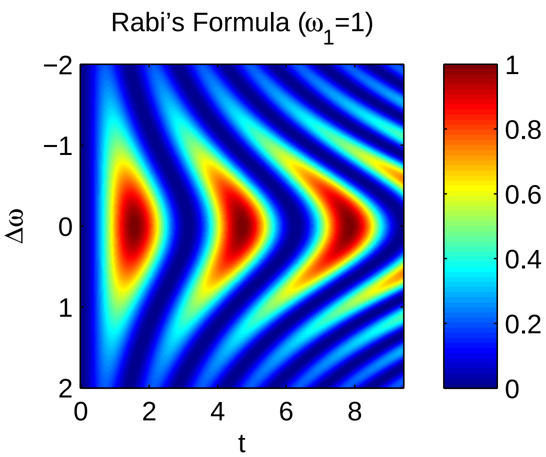
The matrix elements

$$\begin{aligned} \langle 1|\xi_0\rangle &= -\sin(\delta/2) \\ \langle\xi_0|0\rangle &= \cos(\delta/2) \\ \langle 1|\xi_1\rangle &= \cos(\delta/2) \\ \langle\xi_1|0\rangle &= \sin(\delta/2) \end{aligned}$$

after some algebra we can find

$$P_1(t) = |\langle 1|\Psi(t)\rangle|^2 = \frac{\omega_1^2}{\Delta\omega^2+\omega_1^2}\sin\left(\sqrt{\Delta\omega^2+\omega_1^2}t\right)^2$$

```
t,domega=meshgrid(linspace(0,3*pi,401),linspace(-2,2,401))
rabi=1/(1+domega**2)*sin(t*sqrt(domega**2+1))**2
pcolor(t,domega,rabi)
colorbar()
title('Rabi\'s Formula ($\omega_1=1$)')
xlabel('t')
ylabel('$\Delta\omega$')
```



(c) The $\sin^2$ averages out to 1/2, since

$$\sin(x)^2 = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\bar{P}_1 = \frac{1}{2}\frac{\omega_1^2}{\Delta\omega^2+\omega_1^2}$$

Note that this is a Lorentzian lineshape

(d)

$$\omega_{\text{Rabi}} = 2\sqrt{\Delta\omega^2 + \omega_1^2}$$

On resonance (i.e. $\Delta\omega = 0$) the Rabi frequency is $\omega_{\text{Rabi}} = 2A/\hbar$